

PURELY ALGEBRAIC METHOD TO CONSTRUCT TORIC SCHEMES

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ABSTRACT. In this article, we first give some elementary proprieties of monoids and fans, then construct a toric scheme over an arbitrary ring, from a given fan. Using Valutive Criterion, we prove that this scheme is separated and give the sufficient and necessary condition when it is proper. We also study the regularity and logarithmic regularity of it. Finally we study the morphisms of toric schemes induced by the homomorphisms of fans.

INTRODUCTION

The study of toric varieties began in the early 1970's. Till now, people have made great progress in this field. Many applications in algebraic geometry, complex manifolds and number theory etc. have been discovered. Tadao Oda [4] is a comprehensive book on this area. Till now the treatment of toric varieties need more or less the technique of complex analysis. This may hamper its applications in number theory, where we must study algebraic scheme over any base ring, especially over \mathbb{Z} or finite fields. So I am determined to find a purely algebraic approach once and for all.

The keystone of this kind of construction is the concepts of *convex cone* and *concave cone*. In geometry, convex cones are always contained in some affine space, whereas convex cones here are contained in some free abelian group of finite rank, which acts the role of the affine space. For convenience to algebraic treatment, we invent its dual notion *concave cone*. Having these two concepts as bricks, we can easily define fans and toric schemes (over arbitrary rings); and their properties, for example separatedness and properness, easily follow.

We also construct logarithmical structures on toric schemes and prove that when their base rings are regular, they are logarithmically regular. Hence by [1], their singularities can be solved.

In this article, rings, algebras and monoids are all assumed to be commutative and have identity elements. A homomorphism of rings (resp. monoids) is assumed to preserve the identity element. A subring (resp. monoid) is assumed to contain 1 of the total ring (resp. monoid).

1. MONOIDS AND IDEALS

In this section, we list some elementary results about monoids and ideals, which can also be found in many other references, for example, [1].

Lemma 1.1. *A monoid M is integral if and only if $M \rightarrow M^{\text{gp}}$ is injective, where M^{gp} is the Grothendieck group associated with M .*

Definition 1.2. Let M be an integral monoid and P a submonoid of M . If $P^{\text{gp}} \cap M = P$, then we say that P is *full* in M .

Lemma 1.3. *Let M be an integral monoid and S a submonoid of M .*

- (1) $S^{-1}M$ is an integral monoid with Grothendieck group M^{gp} .
- (2) If M is saturated, so is $S^{-1}M$.

Lemma 1.4. *Let M be an integral monoid and N a submonoid of M . Then*

- (1) M/N is also integral.

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(2) *There is an isomorphism of groups:*

$$(M/N)^{\text{gp}} \xrightarrow{\sim} M^{\text{gp}}/N^{\text{gp}}, \quad \frac{\bar{x}}{\bar{y}} \mapsto \frac{\overline{x}}{\overline{y}}.$$

(3) *If M is saturated, so is M/N .*

Lemma 1.5. *Let M be a finitely generated saturated monoid. Then there is a submonoid N of M such that $M = N \times M^*$. Moreover all such submonoids N satisfy the following conditions.*

- (1) $N^* = \{1\}$ and $M^{\text{gp}} = N^{\text{gp}} \times M^*$.
- (2) N is finitely generated and saturated.

Definition 1.6. Let M be a monoid. A subset I of M is called an *ideal* of M if $MI \subseteq I$. An ideal I of M is called a *prime ideal* if its complement $M - I$ is a submonoid of M . We denote by $\text{Spec } M$ the set of prime ideals of M .

For example, \emptyset and $M - M^*$ are prime ideals of M .

For $\mathfrak{p} \in \text{Spec } M$, we define $M_{\mathfrak{p}} := S^{-1}M$, where $S = M - \mathfrak{p}$.

For a homomorphism $h: M \rightarrow N$ of monoids, we have a canonical map $\text{Spec } N \rightarrow \text{Spec } M$, $\mathfrak{p} \mapsto h^{-1}(\mathfrak{p})$.

Theorem 1.7. *Let M be a monoid and \mathfrak{p} a prime ideal of M . Put $N = M - \mathfrak{p}$. Then:*

- (1) $\mathfrak{q} \mapsto N^{-1}\mathfrak{q}$ is a bijective from the set $\{\mathfrak{q} \in \text{Spec } M \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ to $\text{Spec } M_{\mathfrak{p}}$.
- (2) $\mathfrak{P} \mapsto \mathfrak{P} \cap N$ is a bijective from the set $\{\mathfrak{P} \in \text{Spec } M \mid \mathfrak{p} \subseteq \mathfrak{P}\}$ to $\text{Spec } N$ with the inverse map $\mathfrak{q} \mapsto \mathfrak{q} \cup \mathfrak{p}$.

Assume that M is integral, then:

- (3) $(M_{\mathfrak{p}})^* = N^{\text{gp}}$.
- (4) *There is a canonical isomorphism of monoids:*

$$M/N \cong M_{\mathfrak{p}}/(M_{\mathfrak{p}})^*.$$

Hence $(M/N)^ = \{1\}$.*

- (5) *If M is finitely generated, so is N .*

Assume that M is saturated, then:

- (6) N is a saturated, full submonoid of M .
- (7) $M^{\text{gp}}/N^{\text{gp}}$ is torsion free.

Definition 1.8. For a monoid M , we define $\dim(M)$ to be the maximal length of a sequence of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ of M . If such a maximum does not exist, we define $\dim(M) = \infty$.

Lemma 1.9. *Let M be a finitely generated monoid. Then $\text{Spec } M$ is a finite set.*

Lemma 1.10. *Let k be a field and M a finitely generated integral monoid. Assume that M^{gp} is torsion free. Then*

$$\dim(k[M]) = \text{rank}(M^{\text{gp}}).$$

Lemma 1.11. *Let M be a finitely generated integral monoid. Assume that $M^* = \{1\}$ and M^{gp} is torsion free. Then $\dim(M) = 1$ if and only if $M \cong \mathbb{N}$.*

2. ALGEBRAS GENERATED BY MONOIDS

Lemma 2.1. *Let M be a finitely generated saturated monoid with $M^* = \{0\}$. Then M is isomorphic to a submonoid of \mathbb{N}^n for some integer $n \geq 1$.*

Proof. For convenience to iterate, we view M as an additive monoid. We use induction on $r = \text{rank}(M^{\text{gp}})$. If $r = 0$, then $M = \{0\}$; and if $r = 1$, $M \cong \mathbb{N}$.

Now assume that $r > 1$. By Lemma 1.11, $\dim(M) > 1$. Let \mathfrak{p}_1 be a maximal element of the set $\{\mathfrak{p} \in \text{Spec } M \mid \mathfrak{p} \neq M - \{0\}\}$. Put $N_1 = M - \mathfrak{p}_1$. By Theorem 1.7(2), $\dim N_1 = 1$. Hence by Lemma 1.11, $N_1 \cong \mathbb{N}$. Put $N_1 = \mathbb{N} \cdot x$, where $x \in M$.

Let k be a field. Put $A = k[M]$ and $\mathfrak{m} = (M - \{0\}) \cdot A$. By Lemma 1.10,

$$\text{ht}(\mathfrak{m}) = \dim A = r > 1.$$

Let \mathfrak{P} be a minimal prime ideal of A containing x . Then $\text{ht}(\mathfrak{P}) = 1$. We have

$$x \in \mathfrak{p}_2 := \mathfrak{P} \cap M \neq M - \{0\}.$$

Put $N_2 = M - \mathfrak{p}_2$. By Theorem 1.7, for each $i = 1, 2$, M_i/N_i is a finitely generated saturated monoid with $(M_i/N_i)^* = \{0\}$. Obviously

$$\text{rank}(M/N_i)^{\text{gp}} = \text{rank}(M^{\text{gp}}/N_i^{\text{gp}}) < \text{rank}(M^{\text{gp}}) = r.$$

By induction, there is an injective homomorphism $l_i: M_i/N_i \rightarrow \mathbb{N}^{n_i}$ of monoids. Let $\pi_i: M \rightarrow M/N_i$ be the canonical map. Put

$$\sigma: M \rightarrow M/N_1 \oplus M/N_2, \quad x \mapsto (\pi_1(x), \pi_2(x)).$$

Let $\eta \in N_1^{\text{gp}} \cap N_2^{\text{gp}}$. Note that $N_1^{\text{gp}} = \mathbb{Z} \cdot x$. Hence if $\eta \neq 0$, we may assume that $\eta = ax = y - z$, where $a > 0$ and $y, z \in N_2$. Then

$$y = ax + z \in \mathfrak{p}_2,$$

a contradiction. So $N_1^{\text{gp}} \cap N_2^{\text{gp}} = \{0\}$, which infers that the homomorphism

$$\sigma^{\text{gp}}: M^{\text{gp}} \rightarrow (M^{\text{gp}}/N_1^{\text{gp}}) \oplus (M^{\text{gp}}/N_2^{\text{gp}}), \quad \xi \mapsto (\pi_1^{\text{gp}}(\xi), \pi_2^{\text{gp}}(\xi))$$

is an injective, thus σ is an injective. Put $n = n_1 + n_2$. Then

$$l := (l_1 \oplus l_2) \circ \sigma: M \rightarrow \mathbb{N}^n$$

is an injective homomorphism of monoids. □

Applying 2.1 and the results from [2, p.320], we obtain the following theorem.

Theorem 2.2. *Let M be a monoid. Then the following conditions are equivalent:*

- (1) M is isomorphic to a submonoid of \mathbb{N}^r for some integer $r > 0$ and the ring $K[M]$ is integrally closed for some field K .
- (2) M is finitely generated, saturated and $M^* = \{1\}$.
- (3) M is isomorphic to a full submonoid of \mathbb{N}^r for some integer $r > 0$.
- (4) M is isomorphic to a finitely generated submonoid of \mathbb{N}^r for some integer $r > 0$ and for every integrally closed domain D , the ring $D[M]$ is integrally closed.

3. CONVEX CONES AND CONCAVE CONES

All abelian groups and monoid appearing in this section are presumed to be additive.

We fix a finitely generated free abelian group G . Put

$$G^* := \text{Hom}(G, \mathbb{Z}).$$

We identify G^{**} with G .

Definition 3.1.

- (1) A finitely generated saturated submonoid M of G is called a *convex cone* in G if $M^* = \{0\}$ and G/M^{gp} is torsion free.
- (2) A finitely generated saturated submonoid N of G is called a *concave cone* in G if $N^{\text{gp}} = G$ and G/N^* is torsion free.

Lemma 3.2. *Let M be a finitely generated saturated monoid, \mathfrak{p} a prime ideal of M , $N = M - \mathfrak{p}$, P a monoid and $b \in P$. Then there is a $\sigma \in \text{Hom}(M, P)$ such that $\sigma|_N = 0$ and $\sigma(\mathfrak{p}) \subseteq b + P$.*

Proof. Put $M_0 = M/N$. Then M_0 is also a finitely generated saturated monoid such that $M_0^* = \{0\}$ and M_0^{gp} is a free abelian group. By Theorem 2.2, We may regard M_0 as a full submonoid of \mathbb{N}^r for some integer $r > 0$. Then there is a homomorphism $\tau: \mathbb{N}^r \rightarrow P$ of monoids such that

$$\tau(1, 0, \dots, 0) = \tau(0, 1, \dots, 0) = \dots = \tau(0, 0, \dots, 1) = b.$$

Put $\sigma = \tau|_{M_0} \circ \pi: M \rightarrow P$, where $\pi: M \rightarrow M_0$ is the canonical homomorphism. Then σ satisfies the required conditions. \square

Theorem 3.3.

(1) *Let M be a convex cone in G . Then*

$$\check{M} := \{ \sigma \in G^* \mid \forall x \in M, \sigma(x) \geq 0 \}$$

is a concave cone in G^ .*

(2) *Let N be a concave cone in G . Then*

$$\hat{N} := \{ \sigma \in G^* \mid \forall x \in N, \sigma(x) \geq 0 \}$$

is a convex cone in G^ .*

(3) *For any convex cone M in G , $(\check{M})^\wedge = M$.*

(4) *For any concave cone N in G , $(\hat{N})^\vee = N$.*

(5) *Let M be a convex cone in G and $\mathfrak{p} \in \text{Spec } M$. Then $M - \mathfrak{p}$ is a convex cone in G .*

(6) *Let N be a concave cone in G and $\mathfrak{q} \in \text{Spec } N$. Then $N_{\mathfrak{q}}$ is a concave cone in G . Moreover $(N_{\mathfrak{q}})^* = (N - \mathfrak{q})^{\text{gp}}$.*

Proof. (1) Obviously $(\check{M})^{\text{gp}} \subseteq G^*$ and \check{M} is saturated. Assume that M is generated by $x_1, x_2, \dots, x_r \in M - \{0\}$.

Let $\sigma \in G^*$. Then there is an $s \in \mathbb{N}$ such that $s + \sigma(x_i) \geq 0$ for each i . By Lemma 3.2, there is a $\tau_1 \in \text{Hom}(M, \mathbb{N})$ such that

$$\tau_1(M - \{0\}) \subseteq s + \mathbb{N}.$$

Since G/M^{gp} is torsion free, $\tau_1^{\text{gp}}: M^{\text{gp}} \rightarrow \mathbb{Z}$ can be extended to be a homomorphism $\tau: G \rightarrow \mathbb{Z}$. Obviously $\tau, \tau + \sigma \in \check{M}$. Hence $\sigma \in (\check{M})^{\text{gp}}$. Therefore $G^* = (\check{M})^{\text{gp}}$.

It is easy to show that

$$(\check{M})^* = \{ \sigma \in G^* \mid M^{\text{gp}} \subseteq \ker(\sigma) \}.$$

Hence we have

$$(\check{M})^* \cong (G/M^{\text{gp}})^* \cong G/M^{\text{gp}}$$

is torsion free.

Note that

$$\omega: \check{M} \rightarrow \mathbb{N}^r, \quad \sigma \mapsto (\sigma(x_1), \sigma(x_2), \dots, \sigma(x_r))$$

is a homomorphism of monoids. Put $L = \text{Im}(\omega)$ and let $\xi = (\xi_1, \xi_2, \dots, \xi_r) \in L^{\text{gp}} \cap \mathbb{N}^r$. Write ξ as $\xi = \omega(\sigma) - \omega(\tau)$, where $\sigma, \tau \in \check{M}$. Then we have

$$(\sigma - \tau)(x_i) = \sigma(x_i) - \tau(x_i) = \xi_i \in \mathbb{N}.$$

Hence $\sigma - \tau \in \check{M}$, and $\xi = \omega(\sigma - \tau) \in L$. Therefore $L^{\text{gp}} \cap \mathbb{N}^r = L$, i.e., L is a full submonoid of \mathbb{N}^r . By Theorem 2.2, L is finitely generated. So we may let $\tau_1, \tau_2, \dots, \tau_m \in \check{M}$ such that L is generated by $\omega(\tau_1), \omega(\tau_2), \dots, \omega(\tau_m)$. $\tau_1, \tau_2, \dots, \tau_m \in \check{M}$ generated a submonoid L' of \check{M} . Now for any $\sigma \in \check{M}$, there is a $\tau \in L'$ such that $\omega(\sigma) = \omega(\tau)$, i.e., $\sigma - \tau \in (\check{M})^*$. Hence $\check{M} = L' + (\check{M})^*$ is finitely generated. Therefore \check{M} is a concave cone in G^* .

(2) Obviously \hat{N} is saturated and $(\hat{N})^* = \{0\}$. Put $X = \{\sigma \in G^* \mid N^* \subseteq \ker \sigma\}$. Then $(\hat{N})^{\text{gp}} \subseteq X$. Assume that N is generated by x_1, x_2, \dots, x_n , where $x_1, x_2, \dots, x_r \in N - N^*$ and $x_{r+1}, \dots, x_n \in N^*$.

Let $\sigma \in X$. Then there is an $s \in \mathbb{N}$ such that $s + \sigma(x_i) \geq 0$ for each $1 \leq i \leq r$. By Lemma 3.2, there is a $\tau_1 \in \text{Hom}(M, \mathbb{N})$ such that

$$\tau_1(N - N^*) \subseteq s + \mathbb{N}, \quad \tau_1|_{N^*} = 0.$$

Put $\tau = \tau_1^{\text{gp}}$. Then $\tau, \tau + \sigma \in \hat{N}$, i.e., $\sigma \in \hat{N}^{\text{gp}}$. Therefore $\hat{N}^{\text{gp}} = X$ and G^*/\hat{N}^{gp} is torsion free.

Obviously $\omega: \hat{N} \rightarrow \mathbb{N}^r, \sigma \mapsto (\sigma(x_1), \sigma(x_2), \dots, \sigma(x_r))$ is a homomorphism of monoids. Since $\hat{N} \subseteq X$ and $N^{\text{gp}} = G$, ω is an injective. It is easy to show that $\omega(\hat{N})$ is a full submonoid of \mathbb{N}^r . Hence by Theorem 2.2, \hat{N} is finitely generated.

Therefore \hat{N} is a convex cone in G^* .

(3) For any $x \in G$, we have

$$x \in (\check{M})^\wedge \iff \forall \sigma \in \check{M}, \sigma(x) \geq 0.$$

Hence $M \subseteq (\check{M})^\wedge$. On the other hand let $x \in G - M$. If $x \notin M^{\text{gp}}$, as G/M^{gp} is torsion free, there is an element $\sigma \in G^*$ such that $\sigma|_{M^{\text{gp}}} = 0$ and $\sigma(x) < 0$, hence $x \notin (\check{M})^\wedge$. Assume that $x \in M^{\text{gp}}$. By Theorem 2.2, we may regard M as a full submonoid of \mathbb{N}^r for some integer $r > 0$. Let $p_i: \mathbb{Z}^r \rightarrow \mathbb{Z}$ be the i -th projection. Since $M^{\text{gp}} \cap \mathbb{N}^r = M$ and $x \notin M$, there is a projection p_i satisfying that $p_i(x) < 0$. $p_i|_{M^{\text{gp}}}: M^{\text{gp}} \rightarrow \mathbb{Z}$ can be extended to a homomorphism $\sigma \in G^*$. Then $\sigma \in M$ and $\sigma(x) < 0$. Hence $x \notin (\check{M})^\wedge$. Therefore $M = (\check{M})^\wedge$.

(4) For any $x \in G$, we have

$$x \in (\hat{N})^\vee \iff \forall \sigma \in \hat{N}, \sigma(x) \geq 0.$$

Hence $N \subseteq (\hat{N})^\vee$. Let $x \in G - N$. Let $\pi: G \rightarrow G/N^*$ be the canonical homomorphism. By Lemma 1.4, we may regard that $N/N^* \subseteq G/N^*$ and $(N/N^*)^{\text{gp}} = G/N^*$. Obviously $\pi(x) \notin N/N^*$ and $(N/N^*)^* = \{0\}$. By Theorem 2.2, we may regard N/N^* as a full submonoid of \mathbb{N}^r for some integer $r > 0$. Let $p_i: \mathbb{Z}^r \rightarrow \mathbb{Z}$ be the i -th projection. Then there is a projection p_i satisfying $p_i(\pi(x)) < 0$. Put $\sigma = p_i|_{G/N^*} \circ \pi$. Then $\sigma \in \hat{N}$ and $\sigma(x) < 0$. Hence $N = (\hat{N})^\vee$.

(5) and (6) are obvious. \square

Theorem 3.4. *Let M be a convex cone in G and N be a concave cone in G . Let $\mathfrak{p} \in \text{Spec } M$ and $\mathfrak{q} \in \text{Spec } N$.*

(1) *We have*

$$\check{\mathfrak{p}} := \{\sigma \in \check{M} \mid \exists x \in M - \mathfrak{p}, \sigma(x) > 0\} \in \text{Spec } \check{M}$$

and

$$\check{M}_{\check{\mathfrak{p}}} = (M - \mathfrak{p})^\vee.$$

(2) *We have*

$$\hat{\mathfrak{q}} := \{\sigma \in \hat{N} \mid \exists x \in N - \mathfrak{q}, \sigma(x) > 0\} \in \text{Spec } \hat{N}$$

and

$$\hat{N} - \hat{\mathfrak{q}} = (N_{\mathfrak{q}})^\wedge.$$

(3) $(\check{\mathfrak{p}})^\wedge = \mathfrak{p}$.

(4) $(\hat{\mathfrak{q}})^\vee = \mathfrak{q}$.

(5) $\dim M = \dim \check{M}$ and $\dim N = \dim \hat{N}$.

Proof. (1) Obviously $\check{\mathfrak{p}} \in \text{Spec } \check{M}$ and $\check{M}_{\check{\mathfrak{p}}} \subseteq (M - \mathfrak{p})^\vee$. Assume that M is generated by x_1, x_2, \dots, x_n , where $x_1, x_2, \dots, x_r \in \mathfrak{p}$ and $x_{r+1}, \dots, x_n \in M - \mathfrak{p}$. Let $\sigma \in (M - \mathfrak{p})^\vee$. Then there is an $s \in \mathbb{N}$ such that $s + \sigma(x_i) \geq 0$ for all $1 \leq i \leq r$. By Lemma 3.2, there is a $\tau' \in \text{Hom}(M, \mathbb{N})$

such that $\tau'|_{M-\mathfrak{p}} = 0$ and $\tau'(\mathfrak{p}) \subseteq s + \mathbb{N}$. $\tau'^{\text{gp}}: M^{\text{gp}} \rightarrow \mathbb{Z}$ can be extended to a homomorphism $\tau: G \rightarrow \mathbb{Z}$. Then $\tau \in \check{M} - \check{\mathfrak{p}}$ and $\tau + \sigma \in \check{M}$, i.e., $\sigma \in \check{M}_{\check{\mathfrak{p}}}$. Hence $\check{M}_{\check{\mathfrak{p}}} = (M - \mathfrak{p})^\vee$.

(2) Obviously $\hat{\mathfrak{q}} \in \text{Spec } \hat{N}$ and $\hat{N} - \hat{\mathfrak{q}} \subseteq (N_{\mathfrak{q}})^\wedge$. By Theorem 1.7(3), $(N_{\mathfrak{q}})^* = (N - \mathfrak{q})^{\text{gp}}$. Hence for all $\sigma \in (N_{\mathfrak{q}})^\wedge$, $(N - \mathfrak{q})^{\text{gp}} \subseteq \ker \sigma$, i.e., $\sigma \in \hat{N} - \hat{\mathfrak{q}}$. Therefore $\hat{N} - \hat{\mathfrak{q}} = (N_{\mathfrak{q}})^\wedge$.

(3) For any $x \in G$,

$$x \in (\check{\mathfrak{p}})^\wedge \iff x \in M \text{ and } \exists \sigma \in \check{M} - \check{\mathfrak{p}}, \sigma(x) > 0.$$

Hence $(\check{\mathfrak{p}})^\wedge \subseteq \mathfrak{p}$. By Lemma 3.2, there is a $\tau' \in \text{Hom}(M, \mathbb{N})$ such that $\tau'|_{M-\mathfrak{p}} = 0$ and $\tau'(\mathfrak{p}) \subseteq 1 + \mathbb{N}$. $\tau'^{\text{gp}}: M^{\text{gp}} \rightarrow \mathbb{Z}$ can be extended to a homomorphism $\tau: G \rightarrow \mathbb{Z}$. Then $\tau \in \check{M} - \check{\mathfrak{p}}$ and $\tau(x) > 0$ for any $x \in \mathfrak{p}$. Therefore $(\check{\mathfrak{p}})^\wedge = \mathfrak{p}$.

(4) For any $x \in G$,

$$x \in (\hat{\mathfrak{q}})^\vee \iff x \in N \text{ and } \exists \sigma \in \hat{N} - \hat{\mathfrak{q}} = (N_{\mathfrak{q}})^\vee, \sigma(x) > 0.$$

Since $(N_{\mathfrak{q}})^* = (N - \mathfrak{q})^{\text{gp}}$, $(\hat{\mathfrak{q}})^\vee \subseteq \mathfrak{q}$. By Lemma 3.2, there is a $\tau_1 \in \text{Hom}(M, \mathbb{N})$ such that $\tau_1(\mathfrak{q}) \subseteq 1 + \mathbb{N}$ and $\tau_1|_{N-\mathfrak{q}} = 0$. Put $\tau = \tau_1^{\text{gp}}: G \rightarrow \mathbb{Z}$. Then $\tau \in \hat{N} - \hat{\mathfrak{q}}$ and $\tau(x) > 0$ for any $x \in \mathfrak{q}$. Therefore $(\hat{\mathfrak{q}})^\vee = \mathfrak{q}$.

(5) is from (3) and (4). \square

4. TORIC SCHEMES

In this and following sections, we need some knowledge of log structures. [3] and [1] are good references in this field.

First we give the definition of *fan*. Let G be a finitely generated free abelian group.

Definition 4.1. A *fan* in G is a nonempty collection Δ of convex cones in G satisfying the following condition:

- (1) for any $P \in \Delta$ and $\mathfrak{p} \in \text{Spec } M$, $P - \mathfrak{p} \in \Delta$.
- (2) for any $P, Q \in \Delta$, $P \cap Q = P - \mathfrak{p} = Q - \mathfrak{q}$ for some $\mathfrak{p} \in \text{Spec } P$ and $\mathfrak{q} \in \text{Spec } Q$.

The union $|\Delta| := \bigcup_{P \in \Delta} P$ is called the *support* of Δ . Δ is said to be *finite* if Δ is a finite set, and is said to be *complete* if $|\Delta| = G$.

Now we begin to construct the toric scheme from a given fan Δ in G . Let R be a ring. For each $P \in \Delta$, let

$$U_P := \text{Spec } R[\check{P}].$$

Then $\check{P} \rightarrow R[\check{P}]$ generates a log structure on U_P , denoted by \mathcal{M}_P .

Let $P \in \Delta$ and $\mathfrak{p} \in \text{Spec } P$. Put $Q = P - \mathfrak{p}$ and $S = \check{P} - \check{\mathfrak{p}}$. Then S is a finitely generated monoid. Since $\check{Q} = \check{P}_{\check{\mathfrak{p}}}$ by Theorem 3.4(1), we have

$$R[\check{Q}] = R[\check{P}_{\check{\mathfrak{p}}}] \cong S^{-1}R[\check{M}],$$

hence $U_Q \rightarrow U_P$ is an open immersion and $\mathcal{M}_Q = \mathcal{M}_P|_{U_Q}$.

Now let $P, Q \in \Delta$ and let $\mathfrak{p} \in \text{Spec } P, \mathfrak{q} \in \text{Spec } Q$ such that $P \cap Q = P - \mathfrak{p} = Q - \mathfrak{q}$. Then we may regard that

$$U_P \cap U_Q = U_{P \cap Q}.$$

In this way we can glue $\{U_P \mid P \in \Delta\}$ to form a scheme of finite type over R , denoted by $T(\Delta, R)$, which is called a *toric scheme* over R . Obviously $\{\mathcal{M}_P \mid P \in \Delta\}$ can be glued to form a log structure on $T(\Delta, R)$, denoted by $\mathcal{M}(\Delta, R)$.

Let $O := \{0\} \in \Delta$. Then for any $P \in \Delta$, $U_O \subseteq U_P$.

If $R = k$ is a field, by Lemma 1.10, we have $\dim(T(\Delta, k)) = \text{rank}(G)$.

Lemma 4.2. Let $P, Q \in \Delta$. Then the following conditions are equivalent:

- (1) $P \subseteq Q$.

(2) $U_P \subseteq U_Q$.

(3) *There exists a point $x \in U_P \cap U_Q$ such that $\mathfrak{m}_{X,x} \cap \check{P} = \check{P} - \check{P}^*$.*

Proof. (1) \Rightarrow (2) is by the definition of $T(\Delta, R)$.

(2) \Rightarrow (3). Put $A = R[\check{P}]$. Let \mathfrak{m} be a maximal ideal of R . Set $\mathfrak{P} = \mathfrak{m} \cdot A + (\check{P} - \check{P}^*) \cdot A$. Since $A/\mathfrak{P} \cong (R/\mathfrak{m})[\check{P}^*]$ is integral, $\mathfrak{P} \in \text{Spec } A$. \mathfrak{P} correspond to a point x in U_P , which satisfies the required conditions.

(3) \Rightarrow (1). Let $\mathfrak{p} \in \text{Spec } P$ such that $P \cap Q = P - \mathfrak{p}$. Since $x \in U_P \cap U_Q = U_{P \cap Q} = U_{P - \mathfrak{p}}$, we have $\check{P} - \check{P}^* = \mathfrak{m}_{X,x} \cap \check{P} \subseteq \check{\mathfrak{p}}$. Hence $\check{\mathfrak{p}} = \check{P} - \check{P}^*$ and $\mathfrak{p} = (\check{\mathfrak{p}})^\wedge = \emptyset$. Therefore $P = P \cap Q \subseteq Q$. \square

Theorem 4.3. *Let R be a noetherian ring. Then the following conditions are equivalent.*

- (1) $T(\Delta, R)$ is quasi-compact;
- (2) $T(\Delta, R)$ is a noetherian scheme;
- (3) Δ is finite.

Proof. (3) \Rightarrow (2) \Rightarrow (1) is obvious.

(1) \Rightarrow (3). Since $T(\Delta, R)$ is quasi-compact, there are a finite number of convex cones P_1, P_2, \dots, P_n in Δ such that

$$T(\Delta, R) = \bigcup_{i=1}^n U_{P_i}.$$

Put

$$\Delta' = \{ P \in \Delta \mid P \subseteq P_i \text{ for some } i \}.$$

By the definition of fan and Lemma 1.9, Δ' is a finite set.

Let $P \in \Delta$. Since $P \subseteq P$, by Lemma 4.2, there is a point $x \in U_P$ such that $\mathfrak{m}_{X,x} \cap \check{P} = \check{P} - \check{P}^*$. Assume that $x \in U_{P_i}$. Then by Lemma 4.2, $P \subseteq P_i$, i.e., $P \in \Delta'$. Hence $\Delta = \Delta'$ is a finite set. \square

Theorem 4.4. *If R is integral (resp. integrally closed), then $T(\Delta, R)$ is integral (resp. normal).*

Proof. Let $P \in \Delta$. By Lemma 1.5, there is a finitely generated saturated submonoid N of P such that $N^* = \{1\}$ and $\check{P} = N \times P^*$. We have $P^* \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$. Hence

$$R[\check{P}] \cong R[P^*][N] \cong R[\mathbb{Z}^r][N].$$

By Theorem 2.2 if R is integral (resp. integrally closed), so are $R[\mathbb{Z}^r]$ and $R[\mathbb{Z}^r][N]$. \square

Theorem 4.5. *Let A be a R -algebra. Then*

$$T(\Delta, R) \times_R \text{Spec } A \cong T(\Delta, A).$$

In the following, we will use *Valuative Criterion*(see [EGA] II 7.2.3 and 7.2.8) to discuss the separatedness and properness of toric schemes.

Lemma 4.6. *Let K be a field, v a valuation of K , G a finitely generated free abelian additive group, $f: G \rightarrow K^*$ a homomorphism of abelian groups. Put $N := \{x \in G \mid v(f(x)) \geq 0\}$. If $G \neq N$, then*

- (1) *For any $x \in G$, $x \in N$ or $-x \in N$.*
- (2) *N is a concave cone in G .*
- (3) *$\hat{N} \cong \mathbb{N}$, i.e., $\hat{N} = \mathbb{N} \cdot x$ for some $x \in G^*$.*

Proof. (1) and (2) are obvious.

(3). Put $M = \hat{N}$. Then $M \neq \{0\}$ and $\check{M} = N$. By Lemma 1.11, We have only to prove that $\dim M = 1$. Assume that $\dim M \geq 2$. Then there is a $\mathfrak{p} \in \text{Spec } M$ such that $\mathfrak{p} \neq \emptyset$, $M - \{0\}$. Let $x \in M - (\mathfrak{p} \cup \{0\})$ and $y \in \mathfrak{p}$. By Lemma 3.2, there are $\sigma_1, \sigma_2 \in \check{M}$ such that $\sigma_1(x) > 0$, $\sigma_2|_{M - \mathfrak{p}} = 0$ and $\sigma_2(y) > \sigma_1(y)$. Put $\sigma = \sigma_1 - \sigma_2$. Then $\sigma(x) > 0$ and $\sigma(y) < 0$. Hence $\sigma, -\sigma \notin \check{M} = N$. This contradicts (1). \square

Lemma 4.7. *Let K be a field. Let $\varphi: \text{Spec } K \rightarrow T(\Delta, R)$ be a morphism of schemes. Then $\varphi(\text{Spec } K) \subseteq U_O$.*

Lemma 4.8. *$T(\Delta, R)$ is quasi-separated over R .*

Proof. Note that for all $P, Q \in \Delta$, $U_P \cap U_Q = U_{P \cap Q}$ is affine, a fortiori quasi-compact. \square

Theorem 4.9. *$T(\Delta, R)$ is separated over R .*

Proof. Let (A, K, v) be a valuation ring. Let $\varphi: \text{Spec } K \rightarrow T(\Delta, R)$, $\psi: \text{Spec } A \rightarrow \text{Spec } R$ and $\phi_i: \text{Spec } A \rightarrow T(\Delta, R)$ ($i = 1, 2$) be morphisms of schemes which make a commutative diagram.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\varphi} & T(\Delta, R) \\ \downarrow & \nearrow \phi_i & \downarrow \\ \text{Spec } A & \xrightarrow{\psi} & \text{Spec } R \end{array} \quad (4.1)$$

Let \mathfrak{m}_v denote the maximal ideal of A . Let $P_i \in \Delta$ such that $\phi_i(\mathfrak{m}_v) \in U_{P_i}$. Then $\phi_i(\text{Spec } A) \subseteq U_{P_i}$. By Lemma 4.7, $\varphi(\text{Spec } K) \subseteq U_O$. Hence Diagram (4.1) induces a commutative diagram of rings.

$$\begin{array}{ccccc} R & \hookrightarrow & R[\check{P}_i] & \hookrightarrow & R[G^*] \\ \downarrow l & \nearrow g_i & \downarrow f_i & & \\ A & \hookrightarrow & K & & \end{array}$$

Put

$$N := \{x \in G^* \mid v(f(x)) \geq 0\}.$$

If $G^* = N$, then $P := \hat{N} = \{0\}$. If $G^* \neq N$, by the Lemma 4.6, N is a concave in G^* . Then $\check{P}_i \subseteq N$, $\hat{N} \subseteq (\check{P}_i)^\wedge = P_i$. We have

$$P := P_1 \cap P_2 \supseteq \hat{N}.$$

In both cases, we have $P \in \Delta$ and $\check{P}_i \subseteq \check{P} \subseteq \check{N}$. Hence we have $f(R[\check{P}_i]) \subseteq f(R[\check{P}]) \subseteq A$ and $\phi_i(\text{Spec } A) \subseteq U_{\check{P}} \subseteq U_{\check{P}_i}$. So $\phi_1 = \phi_2$. As $T(\Delta, R)$ is quasi-separated, by Valuation Criterion, $T(\Delta, R)$ is separated over R . \square

Theorem 4.10. *$T(\Delta, R)$ is proper over R if and only if Δ is a finite and complete fan.*

Proof. (1) Assume that $T(\Delta, R)$ is proper over R . Let \mathfrak{m} be a maximal ideal of R and set $k = R/\mathfrak{m}$. Then $T(\Delta, k)$ is proper over k , thus is a noetherian scheme. By Theorem 4.3, Δ is finite.

Let x be any element in G . Then there is an element $y \in G$ such that $x = ry$ and $G/\mathbb{Z}y \cong \mathbb{Z}^{n-1}$, where $r \in \mathbb{N}$ and $n = \text{rank}(G)$. Put $M := \mathbb{N}y$, then M is a convex cone in G . Put $N = \check{M}$, $\mathfrak{p} = N - N^*$, $A = k[N]$ and $\mathfrak{P} = \mathfrak{p}A \in \text{Spec } A$. Let K be the quotient field of A . Then there is a valuation ring (B, v) of K/k such that $A \subseteq B$ and $\mathfrak{m}_v \cap A = \mathfrak{P}$. By Lemma 4.6, $N' := \{u \in G^* \mid v(u) \geq 0\}$ is a concave cone in G and $\hat{N}' \cong \mathbb{N}$. Since $N \subseteq N'$, we have $\hat{N}' \subseteq \hat{N} = M$, hence $\hat{N}' = M$. Since $T(\Delta, k)$ is proper over k , B has a unique center ξ on $T(\Delta, k)$. Let $P \in \Delta$ such that $\xi \in U_P$. Then $\check{P} \subseteq N'$, and we have $x \in M = \hat{N}' \subseteq P$. Hence Δ is complete.

(2) Assume that Δ is finite and complete. By Theorem 4.9, $T(\Delta, R)$ is separated over R .

Let (A, K, v) be a valuation ring. Let $\varphi: \text{Spec } K \rightarrow T(\Delta, R)$ and $\psi: \text{Spec } A \rightarrow \text{Spec } R$ be morphisms of schemes which make a commutative diagram.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\varphi} & T(\Delta, R) \\ \downarrow & & \downarrow \\ \text{Spec } A & \xrightarrow{\psi} & \text{Spec } R \end{array} \quad (4.2)$$

By Lemma 4.7, $\varphi(\text{Spec } K) \subseteq U_O$. Hence (4.2) induces a diagram of rings.

$$\begin{array}{ccc} R & \xrightarrow{l} & A \\ \downarrow & & \downarrow \\ R[G^*] & \xrightarrow{f} & K \end{array}$$

Put $N := \{x \in G^* \mid v(f(x)) \geq 0\}$. If $N = G^*$, then $f(R[G^*]) \subseteq A$. Hence $f: R[G^*] \rightarrow A$ induces a morphism $\phi: \text{Spec } A \rightarrow U_O \subseteq T(\Delta, R)$ which make (4.3) commutative.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\varphi} & T(\Delta, R) \\ \downarrow & \searrow \phi & \downarrow \\ \text{Spec } A & \xrightarrow{\psi} & \text{Spec } R \end{array} \quad (4.3)$$

So we may assume that $N \neq G^*$. By Lemma 4.6, N is a concave cone in G^* and $\hat{N} = \mathbb{N} \cdot \varepsilon$ for some $\varepsilon \in G$. Since Δ is complete, there is a $P \in \Delta$ containing ε . Then $\check{P} \subseteq N$. We have $f(R[\check{P}]) \subseteq A$. Hence

$$g := f|_{R[\check{P}]}: R[\check{P}] \rightarrow A$$

induces a morphism

$$\phi: \text{Spec } A \rightarrow U_P \subseteq T(\Delta, R),$$

which make (4.3) commutative. By Valutive Criterion, $T(\Delta, R)$ is proper over R . \square

5. REGULARITY AND LOGARITHMICAL REGULARITY

Now we study the regularity of Toric Schemes.

Lemma 5.1. *Let R be a noetherian ring, M a finitely generated saturated monoid such that M^{gp} is torsion-free. Then $R[M]$ is regular if and only if R is regular and $M \cong \mathbb{Z}^r \times \mathbb{N}^s$ for some $r, s \in \mathbb{N}$.*

Proof. Put $A = R[M]$. Obviously if R is regular and $M \cong \mathbb{Z}^r \times \mathbb{N}^s$, A is regular.

Now assume that A is regular. Let \mathfrak{p} be any prime ideal of R . Then $\mathfrak{P} := \mathfrak{p} \cdot A$ is a prime ideal of A . Since A is flat over R , $A_{\mathfrak{P}}$ is flat over $R_{\mathfrak{p}}$. By [6, Theorem 23.7], $R_{\mathfrak{p}}$ is regular. Hence R is a regular ring.

By Lemma 1.5, $M \cong \mathbb{Z}^r \times N$, where N is a finitely generated saturated monoid with $N^* = \{1\}$. Put $A' = R[\mathbb{Z}^r]$. Let \mathfrak{p} be a minimal prime ideal of A' . Since R is regular, so is A' and $A'_{\mathfrak{p}}$. Hence $K := A'_{\mathfrak{p}}$ is a field. As $A'[N] \cong R[M]$ is regular, so is its localization $K[N] \cong S^{-1}(A'[N])$, where $S = A' - \mathfrak{p}$. Put $B = K[N]$ and $\mathfrak{m} = (N - \{1\}) \cdot B$. Set

$$T = N - \{1\} \cup \{x \cdot y \mid x, y \in N - \{1\}\}.$$

Then N can be generated by T . Obviously $K \cdot T$ is a K -linear subspace of B with dimension $= |T|$. Put $\mathfrak{m}' = \mathfrak{m}B_{\mathfrak{m}}$. Then we have

$$\mathfrak{m}'/\mathfrak{m}'^2 \cong \mathfrak{m}/\mathfrak{m}^2 \cong K \cdot T.$$

By Lemma 1.10, $\dim B_{\mathfrak{m}'} = \text{rank}(N^{\text{gp}})$. Since $B_{\mathfrak{m}'}$ is a regular local ring, we have

$$|T| = \dim_K(\mathfrak{m}'/\mathfrak{m}'^2) = \dim B_{\mathfrak{m}'} = \text{rank}(N^{\text{gp}}).$$

Put $T = \{x_1, x_2, \dots, x_s\}$, then the following homomorphism

$$\mathbb{Z}^s \rightarrow N^{\text{gp}}, \quad (a_1, a_2, \dots, a_s) \mapsto x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}$$

is an isomorphism. Hence $N \cong \mathbb{N}^s$, i.e., $M \cong \mathbb{Z}^r \times \mathbb{N}^s$. \square

Theorem 5.2. *$T(\Delta, R)$ is a regular scheme if and only if R is regular and for each $P \in \Delta$, $P \cong \mathbb{N}^r$ for some $r \in \mathbb{N}$.*

Next, we study the logarithmical regularity of Toric Schemes. First, we give the definition of logarithmical regularity introduced in [1, p.1075-1076].

Definition 5.3. Let (X, \mathcal{M}) be a log scheme. We say (X, \mathcal{M}) is *logarithmically regular* at a point $x \in X$, if the following two conditions are satisfied. Let $I(x, \mathcal{M})$ be the ideal of $\mathcal{O}_{X,x}$ generated by the image $\mathcal{M}_x - \mathcal{O}_{X,x}^*$.

- (1) $\mathcal{O}_{X,x}/I(x, \mathcal{M})$ is a regular local ring.
- (2) $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/I(x, \mathcal{M})) + \text{rank}(\mathcal{M}_x^{\text{gp}}/\mathcal{O}_{X,x}^*)$.

We say (X, \mathcal{M}) is *logarithmically regular* if (X, \mathcal{M}) is logarithmically regular at all $x \in X$.

Lemma 5.4. Let R be a noetherian local ring with maximal ideal \mathfrak{m} , M a finitely generated saturated monoid with $M^* = \{1\}$. Put $A = R[M]$ and $\mathfrak{m}' = \mathfrak{m} \cdot A + (M - \{1\}) \cdot A$. Then \mathfrak{m}' is a maximal ideal of A with $\text{ht}(\mathfrak{m}') = \dim(R) + \text{rank}(M^{\text{gp}})$.

Proof. Obviously A is flat over R and $\mathfrak{m}' \cap R = \mathfrak{m}$. We have

$$\begin{aligned} \text{ht}(\mathfrak{m}') &= \text{ht}(\mathfrak{m}) + \text{ht}(\mathfrak{m}'/\mathfrak{m}A) \quad (\text{by [5], (13.B), Theorem 19}) \\ &= \dim(R) + \dim(k[M]) \quad (\text{put } k = R/\mathfrak{m}) \\ &= \dim(R) + \text{rank}(M^{\text{gp}}) \quad (\text{by Lemma 1.10}) \quad \square \end{aligned}$$

Theorem 5.5. Let R be a regular ring. Then $(T(\Delta, R), \mathcal{M}(\Delta, R))$ is logarithmically regular.

Proof. Put $X = T(\Delta, R)$ and $\mathcal{M} = \mathcal{M}(\Delta, R)$. We use the notations introduced in Definition 5.3. Let x be any point of X . Assume that $x \in U_P$, where $P \in \Delta$. Put $M = \check{P}$, $A = \mathcal{O}_{X,x}$ and $I = I(x, \mathcal{M})$. Then $U_P = \text{Spec } R[M]$ and x corresponds to a prime ideal \mathfrak{P} of $R[M]$. Thus $A \cong R[M]_{\mathfrak{P}}$. Put $\mathfrak{p} = M \cap \mathfrak{P}$. Then I corresponds to the ideal $\mathfrak{p} \cdot R[M]_{\mathfrak{P}}$ of $R[M]_{\mathfrak{P}}$. By Lemma 1.5, there is an integer $r \geq 0$ and a finitely generated saturated monoid N such that $N^* = \{1\}$ and $M_{\mathfrak{p}} \cong N \times \mathbb{Z}^r$. Then

$$\begin{aligned} A &\cong (R[M_{\mathfrak{p}}])_{\mathfrak{P}_1} \\ &\cong (R_1[N])_{\mathfrak{P}_2} \quad (\text{put } R_1 = R[\mathbb{Z}^r]) \\ &\cong (R_2[N])_{\mathfrak{P}_3} \quad (\text{put } R_2 = (R_1)_{\mathfrak{P}_2 \cap R_1}) \end{aligned}$$

Obviously R_2 is a regular local ring with maximal ideal $\mathfrak{m}_2 = (\mathfrak{P}_2 \cap R_1) \cdot R_2$. Put $A' = R_2[N]$, $\mathfrak{m}' = \mathfrak{m}_2 \cdot A' + (N - \{1\}) \cdot A'$ and $B = A'_{\mathfrak{P}_3}$. Since $\mathfrak{m}' \subseteq \mathfrak{P}_3$ and \mathfrak{m}' is a maximal ideal of A' , we have $\mathfrak{P}_3 = \mathfrak{m}'$. Note that I corresponds to the ideal $(N - \{1\}) \cdot B$ of B . Hence $A/I \cong R_2$ is a regular local ring. We have

$$\begin{aligned} \dim(A) &= \text{ht}(\mathfrak{m}') \\ &= \dim(R_2) + \text{rank}(N^{\text{gp}}) \quad (\text{by Lemma 5.4}) \\ &= \dim(A/I) + \text{rank}(\mathcal{M}_x^{\text{gp}}/\mathcal{M}_x^*) \end{aligned}$$

By the definition, (X, \mathcal{M}) is regular at x . \square

6. MORPHISMS OF TORIC SCHEMES

Definition 6.1. Let (G, Δ) and (G', Δ') be two fans. A homomorphism $\varphi: (G, \Delta) \rightarrow (G', \Delta')$ of fans is a homomorphism $\varphi: G \rightarrow G'$ of groups satisfying that: for each $P \in \Delta$ there exists a $P' \in \Delta'$ such that $\varphi(P) \subseteq P'$.

In the following, we let R be a ring.

Theorem 6.2. *Let $\varphi: (G, \Delta) \rightarrow (G', \Delta')$ be a homomorphism of fans. Then φ gives rise to a morphism*

$$\varphi_*: (T(\Delta, R), \mathcal{M}(\Delta, R)) \rightarrow (T(\Delta', R), \mathcal{M}(\Delta', R))$$

of log schemes over R .

Theorem 6.3. *Let $\varphi: (G, \Delta) \rightarrow (G', \Delta')$ be a homomorphism of fans. Then for any $P' \in \Delta'$,*

$$\varphi_*^{-1}(U_{P'}) = \bigcup_{\substack{P \in \Delta \\ \varphi(P) \subseteq P'}} U_P.$$

Proof. Put $X = T(\Delta, R)$, $X' = T(\Delta', R)$ and $f = \varphi_*: X \rightarrow X'$. Let $x \in f^{-1}(U_{P'})$. We may assume that $x \in U_{P_1}$, where $P_1 \in \Delta$. Then there is a $P'' \in \Delta'$ such that $\varphi(P_1) \subseteq P''$. We have $\mathfrak{p}' \in \text{Spec } P'$ and $\mathfrak{p}'' \in \text{Spec } P''$ such that

$$P' \cap P'' = P' - \mathfrak{p}' = P'' - \mathfrak{p}''.$$

Put $Q_1 = \check{P}_1$, $Q' = \check{P}'$, $Q'' = \check{P}''$, $\mathfrak{q}' = \check{\mathfrak{p}}'$ and $\mathfrak{q}'' = \check{\mathfrak{p}}''$. x corresponds to a prime ideal \mathfrak{P} of $R[Q_1]$. Put $\mathfrak{q}_1 = \mathfrak{P} \cap Q_1$, $\psi = \varphi|_{P_1}: P_1 \rightarrow P''$ and $\phi = \varphi^*|_{Q''}: Q'' \rightarrow Q_1$. Since

$$\varphi(x) \in U_{P'} \cap U_{P''} = U_{P' \cap P''} = U_{P'' - \mathfrak{p}''},$$

we have a natural homomorphism of R -algebras: $R[Q''_{\mathfrak{q}''}] \rightarrow R[Q_1]_{\mathfrak{P}}$. Using the following commutative diagram

$$\begin{array}{ccccc} Q'' & \hookrightarrow & Q''_{\mathfrak{q}''} & \hookrightarrow & R[Q''_{\mathfrak{q}''}] \\ \downarrow \phi & & & & \downarrow \\ Q_1 & \xrightarrow{\quad} & & & R[Q_1]_{\mathfrak{P}} \end{array}$$

we obtain $\phi^{-1}(\mathfrak{q}_1) \subseteq \mathfrak{q}''$. Hence $\mathfrak{p} := \psi^{-1}(\mathfrak{q}'') \subseteq \hat{\mathfrak{q}}_1$. Put $P = P_1 - \mathfrak{p}$. Then $\varphi(P) \subseteq P'' - \mathfrak{p}'' \subseteq P'$. Since $\mathfrak{q}_1 \subseteq \check{\mathfrak{p}}$, $x \in U_P \subseteq f^{-1}(U_{P'})$. \square

Theorem 6.4. *Let $\varphi: (G, \Delta) \rightarrow (G', \Delta')$ be a homomorphism of fans. Then $\varphi_*: T(\Delta, R) \rightarrow T(\Delta', R)$ is proper if and only if for each $P' \in \Delta'$, the set*

$$\Delta_{P'} := \{P \in \Delta \mid \varphi(P) \subseteq P'\}$$

is finite and

$$\varphi^{-1}(P') = \bigcap_{P \in \Delta_{P'}} P.$$

Proof. By Theorem 6.3 and Lemma 1.9, we may assume that Δ' is finite.

(1) Assume that φ_* is a proper morphism. Let \mathfrak{m} be a maximal ideal of R and set $k = R/\mathfrak{m}$. Then

$$X := T(\Delta, k) \cong T(\Delta, R) \times_R \text{Spec } k,$$

$$X' := T(\Delta', k) \cong T(\Delta', R) \times_R \text{Spec } k,$$

and the morphism

$$f := \varphi_* \times_R \text{id}: X \rightarrow X'$$

is proper. As X' is a noetherian scheme by Theorem 4.3, so is X . Hence Δ is a finite set.

Let $P' \in \Delta'$ and $u \in \varphi^{-1}(P')$ be any elements. Then there is an $e \in G$ such that $u = ae$ and $G/\mathbb{Z}e \cong \mathbb{Z}^{n-1}$, where $a \in \mathbb{N}$ and $n = \text{rank}(G)$. $M := \mathbb{N}e$ is a convex cone in G . Put $N = \check{M}$, $\mathfrak{p} = N - N^*$, $A = k[N]$ and $\mathfrak{P} = \mathfrak{p}A \in \text{Spec } A$. Let K be the quotient field of A . Then there is a valuation ring (B, v) of K/k such that $A \subseteq B$ and $\mathfrak{m}_v \cap A = \mathfrak{P}$. So $N = \{u \in G^* \mid v(u) \geq 0\}$. As P' is saturated and $\varphi(u) = a \cdot \varphi(e) \in P'$, we have $\varphi(e) \in P'$, hence

$\varphi(M) \subseteq P'$. φ induces a homomorphism $k[\check{P}'] \rightarrow k[N]$, and the following commutative diagram of rings

$$\begin{array}{ccccc} k[\check{P}'] & \longrightarrow & k[N] & \hookrightarrow & B \\ & & \downarrow & & \downarrow \\ & & k[G^*] & \hookrightarrow & K \end{array}$$

induces a commutative diagram of schemes

$$\begin{array}{ccccc} \text{Spec } K & \longrightarrow & U_O & \hookrightarrow & X \\ \downarrow & & & & \downarrow f \\ \text{Spec } B & \longrightarrow & U_{P'} & \hookrightarrow & X' \end{array}$$

By Valutive Criterion, there is a morphism $g: \text{Spec } B \rightarrow X$ of schemes which make a diagram

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & X \\ \downarrow & \nearrow g & \downarrow f \\ \text{Spec } B & \longrightarrow & X' \end{array}$$

Since $f(g(m_v)) \in U_{P'}$, by Theorem 6.3, there exists a $P \in \Delta$ such that $g(m_v) \in U_P$ and $\varphi(P) \subseteq P'$. As $k[\check{P}] \subseteq B$, we have $\check{P} \subseteq N$. Hence $u \in M = \hat{N} \subseteq P$.

(2) Assume that for each $P' \in \Delta'$, $\Delta_{P'}$ is finite and

$$\varphi^{-1}(P') = \bigcap_{P \in \Delta_{P'}} P.$$

Let $X := T(\Delta, R)$, $X' := T(\Delta', R)$ and $f = \varphi_*: X \rightarrow Y$. By Theorem 4.9, f is separated. Let (A, K, v) be a valuation ring. Let $\alpha: \text{Spec } K \rightarrow X$ and $\beta: \text{Spec } A \rightarrow X'$ be the morphisms of schemes which make a commutative diagram.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow f \\ \text{Spec } A & \xrightarrow{\beta} & X' \end{array} \quad (6.1)$$

Assume that $\beta(\mathfrak{m}_v) \subseteq U_{P'}$, where $P' \in \Delta'$. By Lemma 4.7, $\varphi(\text{Spec } K) \subseteq U_O$. Obviously $f(U_O) \subseteq U_{O'} \subseteq U_{P'}$. Then (6.1) induces a commutative diagram of rings

$$\begin{array}{ccc} R[\check{P}'] & \xrightarrow{\iota} & A \\ \downarrow & & \downarrow \\ R[G^*] & \xrightarrow{\delta} & K \end{array}$$

Put $N = \{x \in G^* \mid v(\delta(x)) \geq 0\}$. If $N = G^*$, then $\delta(R[G^*]) \subseteq A$ and $\delta: R[G^*] \rightarrow A$ induces a morphism $g: \text{Spec } A \rightarrow U_O \subseteq X$ which make a commutative diagram.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow g & \downarrow f \\ \text{Spec } A & \xrightarrow{\beta} & X' \end{array} \quad (6.2)$$

So we may assume that $N \neq G^*$. By Lemma 4.6, N is a concave cone in G^* and $\hat{N} = \mathbb{N} \cdot e$ for some $e \in G$. As $\varphi^*(\check{P}') \subseteq N$, we have $e \in \hat{N} \subseteq \varphi^{-1}(P')$. By the assumption, there is a $P \in \Delta$ such that $e \in P \subseteq \varphi^{-1}(P')$. So $\check{P} \subseteq N$, and we have $\delta(R[\check{P}]) \subseteq A$. Hence

$$\delta' := \delta|_{R[\check{P}]}: R[\check{P}] \rightarrow A$$

induces a morphism $g: \text{Spec } A \rightarrow \text{U}_P \subseteq X$ which make (6.2) commutative. By Valuative Criterion, f is a proper morphism. \square

Theorem 6.5. *Let k be a field. Let $\varphi: (G, \Delta) \rightarrow (G', \Delta')$ be a homomorphism of fans. Then $\varphi_*: \text{T}(\Delta, k) \rightarrow \text{T}(\Delta', k)$ is birational if and only if $\varphi: G \rightarrow G'$ is an isomorphism of groups.*

Proof. Obviously we have only to prove that if φ_* is birational, then φ is an isomorphism. Put $H = G^*$, $H' = G'^*$ and $\psi = \varphi^*: H' \rightarrow H$. Since the homomorphism $\phi: k[H'] \rightarrow k[H]$ induced by ψ is an injective, so is ψ . So we may regard H' as a subgroup of H . As φ_* is birational, $k[H']$ and $k[H]$ have the same quotient field, denoted by K . We have

$$\text{rank}(H) = \dim \text{T}(\Delta, k) = \dim \text{T}(\Delta', k) = \text{rank}(H').$$

Hence H/H' is a finite group and $k[H]$ is a finite integral extension of $k[H']$. By Theorem 4.4, $k[H']$ is an integral closed integral domain. Hence $k[H'] = k[H]$, i.e., $H' = H$. Therefore $\varphi: G \rightarrow G'$ is an isomorphism. \square

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